JOURNAL OF APPROXIMATION THEORY 65, 73–89 (1991)

# Module and Vector Space Bases for Spline Spaces

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*Communicated by Charles K. Chui*

Received March 1, 1989

For $\Delta$, a triangulated $d$-dimensional region in $R^d$, let $S^r_m(\Delta)$ be the vector space of all $C^r$ functions $F$ on $\Delta$ such that for any simplex $\sigma \in \Delta$, $F|_\sigma$ is a polynomial of degree at most $m$. $S^r_m(\Delta)$ is the often studied vector space of splines on $\Delta$ of degree $m$ and smoothness $r$. We define $S^r(\Delta) = \bigcup_m S^r_m(\Delta)$. $S^r(\Delta)$ is a module over the polynomial ring $\mathbb{R}[x_1, ..., x_d]$. In certain cases a module basis for $S^r(\Delta)$ provides vector space bases for the corresponding $S^r_m(\Delta)$ via simple linear algebra. In this work we examine that relationship and consider techniques for finding module bases of spaces $S^r(\Delta)$.

A basis for $S^r(\Delta)$ is reduced if every element $F$ in $S^r(\Delta)$ can be represented using only basis elements of degree less than the degree of $F$. We show the relationship between the dimension of the spaces $S^r_m(\Delta)$ and the degrees of the reduced basis elements of $S^r(\Delta)$. Ths result leads to techniques for finding module bases. These techniques are used to find module bases for spline spaces on cross-cut grids.


## 1. INTRODUCTION AND OUTLINE

Let $D \subset \mathbb{R}^d$ be a connected $d$-dimensional domain and $\Delta$ a finite $d$-dimensional complex that subdivides $D$ (e.g., let $\Delta$ be a simplicial complex). For technical convenience we assume that $\Delta$ is pure, i.e., that each maximal face of $\Delta$ is $d$-dimensional. When only one complex $\Delta$ is given for a particular domain $D$ we often use $\Delta$ to denote the underlying domain $D$. For any non-negative integers $r$ and $m$ define $S^r_m(\Delta)$ to be the set of all piecewise polynomial functions on $\Delta$ of degree at most $m$ and which are smooth of order $r$. Precisely, $S^r_m(\Delta)$ is the set of all functions $F: \Delta \to \mathbb{R}$ such that $F|_\sigma$ is a polynomial of degree at most $m$ for each simplex $\sigma \in \Delta$ and $F$ is continuously differentiable of order $r$. Such a function $F$ is a *spline* or *finite element*. The space $S^r_m(\Delta)$ is a vector space over $\mathbb{R}$.

* Supported in part by NSF Grants DMS-8403225 and DMS-8703370 at Cornell University.



0021-9045/91 \$3.00

Splines are used most commonly to approximate functions. Traditionally they have been used by numerical analysts for approximating solutions to differential equations. More recently splines have played an important role in the creation of computer graphics.

It is of interest to find bases for the spaces $S^r_m(\Delta)$. Recently, work has been done to establish the dimension and bases for $S^r_m(\Delta)$ for various $m$, $r$, and $\Delta$ [1, 2, 10, 12, 13, 16].

We define $S^r(\Delta) = \bigcup_m S^r_m(\Delta)$. $S^r(\Delta)$ is a module over the polynomial ring $\mathbb{R}[x_1, ..., x_d]$ [4]. It can be insightful to consider this algebraic structure instead of the vector spaces. In certain cases a module basis for $S^r(\Delta)$ provides vector space bases for the corresponding $S^r_m(\Delta)$ via simple linear algebra. In this work we examine that relationship and consider techniques for finding module bases of spaces $S^r(\Delta)$. Section 2 gives some general considerations about the relationship between the vector spaces and the modules. In Section 3 the special case in which $S^r(\Delta)$ is a graded module is examined and in Section 4 this work is extended to when $\Delta$ is a cross-cut grid in 2-dimensions. Section 5 concludes with some additional remarks and extensions.

## 2. Vector Space Bases and Reduced Free Module Bases

Throughout most of this work we are concerned with spaces $S^r(\Delta)$ that are free modules, i.e., a module that has a basis whose elements are linerly independent. The *rank* of a free module is the number of elements in its free basis. When $S^r(\Delta)$ is free its rank is $f_d$, the number of $d$-faces in $\Delta$ [11]. A basis $\{b_i\}$ is *reduced* if $F \in S^r(\Delta)$ and $\deg F = t$ then $F = \sum \lambda_i b_i$ such that $\deg \lambda_i b_i \leqslant t$.

In this section we examine the relation between reduced free bases for $S^r(\Delta)$ and vector space bases for $S^r_m(\Delta)$. Suppose that $B = \{b_1, ..., b_n\}$ is a reduced module basis for the module $A$, with degree $b_i = d_i$. Then the vector space $A_m = \{a \in A \mid \deg a \leqslant m\}$ has basis $\{b_i x^j y^k \mid d_i + j + k \leqslant m\}$. This is clear since if $a \in A$ and $\deg a \leqslant m$ then

$$a = \sum_{d_i + j + k \leqslant m} \alpha_{ijk} x^j y^k b_i,$$

where $\alpha_{ijk} \in \mathbb{R}$.

We next use combinatorial arguments to determine some properties of the dimensions and the bases for the module $S^r(\Delta)$. Let $B = \{b_1, b_2, ..., b_v\}$ be a reduced free basis of $S^r(\Delta)$. Define $g_i = |\{b_j : \deg b_j = i\}|$, for $i = 0, 1, 2, ...$, i.e., $g_i$ is the number of basis elements of degree $i$. By the definition of reduced free bases any $F \in S^r_m(\Delta)$ can be expressed uniquely as

$F = \sum \lambda_j b_j$ such that $\deg \lambda_j b_j \leqslant m$, i.e., $\deg \lambda_j \leqslant (m - \deg b_j)$. Thus the dimension of $S^r(\Delta)$ is given by

$$\dim S_m^r(\Delta) = \sum_{i=0}^{m} g_i \binom{m-i+d}{d}. \tag{2.1}$$

Define $D^i(m) = D^i(\dim S_m^r(\Delta))$ to be the $i$th difference of $\dim S_m^r(\Delta)$ with respect to $m$, that is, $D^0(m) = \dim S_m^r(\Delta)$, and recursively define $D^{i+1}(m) = D^i(m) - D^i(m-1)$. From Eq. (2.1) we get

$$\begin{aligned}
D^0(m) &= \sum_{i=0}^{m} g_i \binom{m-i+d}{d} \\
D^1(m) &= \sum_{i=0}^{m} g_i \binom{m-i+d-1}{d-1} \\
&\vdots \\
D^j(m) &= \sum_{i=0}^{m} g_i \binom{m-i+d-2}{d-j} \\
&\vdots \\
D^d(m) &= \sum_{i=0}^{m} g_i \\
D^{d+1}(m) &= g_m.
\end{aligned}$$

For $m < 0$, $\dim S_m^r(\Delta) = 0$. We can extend these differences to the cases where $m < 0$ by defining the empty sum to be 0. Consequently, all the $g_i$ values for $i = 0, 1, \ldots$ are given by the $(d+1)$st differences.

THEOREM 2.3. *For any $\Delta$, if $S^r(\Delta)$ has a free reduced basis then the degrees of the reduced basis elements are given by the $(d+1)$st differences of* $\dim S_m^r(\Delta)$.

COROLLARY 2.4. *For any space $S^r(\Delta)$, if the $(d+1)$st differences of* $\dim S_m^r(\Delta)$ *are not all greater than or equal to zero then $S^r(\Delta)$ does not have a reduced free basis.*

EXAMPLE A. $d$-dimensional complex $\Delta$ is *stacked* if it has no interior faces of dimension less than $d-1$. For an example, see Fig. 1a. As a trivial example, we note that for $d = 1$ any complex is stacked. In this case it is relatively easy to construct a basis for the $A$-module $S^r(\Delta)$ and hence to

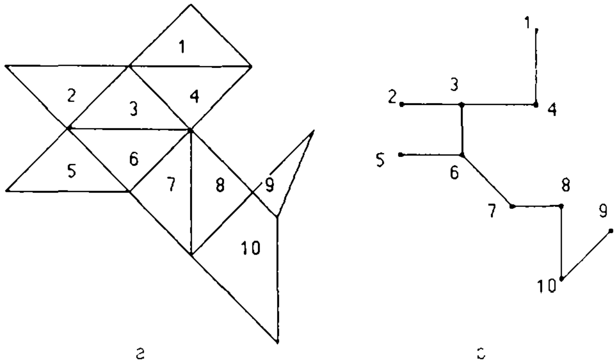


FIG. 1. A stacked complex and its associated graph.

construct a basis for the vector spaces $S^r_m(\Delta)$. The dimension of $S^r_m(\Delta)$, when $\Delta$ is stacked, is given by

$$\dim S^r_m(\Delta) = \binom{m+d}{d} + (f_d - 1)\binom{m-r-1+d}{d}, \tag{2.5}$$

where $f_d$ is the number of $d$-faces in $\Delta$. Taking $d+1$ differences we find that there will be one basis element of degree 0 and $f_d - 1$ of degree $r+1$.

Define the dual graph, $G = (V, E)$, of the complex as follows. For each $\sigma \in \Delta_d$, associate a vertex $v_\sigma \in V$ and for each $\tau = \sigma_1 \cap \sigma_2$ in $\Delta^0_{d-1}$, there is an edge $(v_{\sigma_1}, v_{\sigma_2}) \in E$. For a stacked complex $\Delta$ the associated graph $G$ will be a tree. Figure 1 shows a stacked complex $\Delta$ the associated graph. Choose any vertex $v' \in V$ to be the *root* of $G$. We use the structure of $G$ to construct a basis for $S^r(\Delta)$.

For each vertex $v \in V$, let $v_p$ be the (unique) *parent* of $v$ in $G$, i.e., the adjacent vertex in $G$ that is closer to the root, $v'$. Given two vertices $u$ and $v$, if there is a path of edges joining them which does not contain the root, then $u$ is a *descendent* of $v$ if $v$ is closer to the root than $u$. With each vertex, $v \in V$, associate the affine form $l_v$ that vanishes on the $(d-1)$-simplex of $\Delta$ which corresponds to the edge $(v, v_p)$ in the graph. For the root vertex, $v'$, let $l_{v'} = 1$.

For each $v \in V$, define $b'_v$ to have the value $l_v^{r+1}$ on $v$ and on all its descendents and to have the value 0 on all other vertices. For instance, $b'_{v'}$ is 1 on each vertex. These $b'_v$ correspond to piecewise polynomials $b_v$ on $\Delta$. It is clear that these $b_v$ are actually in $S^r(\Delta)$ and, in fact, $\{b_v : v \in vV\}$ is a basis for the module $S^r(\Delta)$.

Outside the simple stacked case just discussed it is not usually possible to simply observe a basis. However, if the degrees of the module bases are known then the module basis elements can be found via elementary linear algebra methods. The following general result is useful.

PROPOSITION 2.6. *Let $M$ be a free $R$-module with basis $B$. For any subset $B_i \subset B$, if $B_i'$ is another free basis for $\langle B_i \rangle$ then $(B - B_i) \cup B_i'$ is a basis for $M$.*

Given the degrees of the basis elements, we find basis elements in the order of their degree from the least degree up. We proceed by subsequently finding bases of $S_m^r(\Delta)$ for the $m$ for which we known there will be reduced basis elements. First we develop the linear algebra for these computations.

THEOREM 2.7 [4]. *Given a $\Delta$ which is strongly connected and has strongly connected links, let $F$ be a piecewise polynomial function such that $F|_\sigma$ is a polynomial of degree $\leqslant m$ for each $\sigma \in \Delta$. For $r \geqslant 0$, $F \in S_m^r(\Delta)$ if and only if for each pair $\sigma$, $\gamma$ of adjacent $d$-simplices in $\Delta$ we have, if $\tau = \sigma \cap \gamma$ is of dimension $d-1$ and $l$ is a nontrivial affine form which vanishes on $\tau$, $l^{r+1} \mid (F|_\sigma - F|_\gamma)$.*

*Proof.* This result was used and proved in [4 and 18]. ∎

Thus we may consider the piecewise polynomial $F = (f_1, ..., f_k) \in S^r(\Delta)$, where $k$ is the number of $d$-faces in $\Delta$, as the solution to the following linear equations over the polynomial ring. For all $\sigma_i \in \Delta_d$, let $f_i = F|_{\sigma_i}$, and when $\tau = \sigma_i \cap \sigma_j$ has dimension $d-1$ let $l_{ij}$ be a nontrivial affine form that spans $\tau$. For each such $\tau$ we get an equation

$$f_i - f_j + g_{ij} l_{ij}^{r+1} = 0, \tag{2.8}$$

where $g_{ij}$ is a polynomial. If we wish to find solutions $F \in S_m^r(\Delta)$ for some specific $m$ then we can make a set of linear equations over $\mathbb{R}$ to replace (2.8) as follows. In order for the polynomial relation (2.8) to hold it must hold on the coefficients of each monomial $x_1^{r1} x_2^{r2} \cdots x_d^{rd}$, for all $r1 + \cdots + rd \leqslant r$. Let the matrix of these relations be $A_m$; then $S_m^r(\Delta)$ is the null space of $A_m$.

EXAMPLE. Consider the 2-complex consisting of two 2-simplexes that meet along the line $x = 0$. $(f_1, f_2) \in S^r(\Delta)$ if and only if the $f_i$ are polynomials, and there exists a polynomial $g$ such that

$$f_1 - f_2 + g y^{r+1} = 0.$$

Further, let $f_i = \sum a_{ijk} x^j y^k$; then $(f_1, f_2) \in S_m^r(\Delta)$ if and only if $a_{ijk} = 0$ for all $j + k \geqslant m$ and there exists a $g = \sum b_{jk} x^j y^k$ such that for all $j, k$ where $j + k \leqslant m$,

$$a_{1jk} - a_{2jk} + b_{j(k-r-1)} = 0.$$

Note that the polynomial $g$ only has terms with $j + k \leqslant m - r - 1$.

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We proceed with the discussion about how to find a reduced basis. Suppose that the number of elements of degree $m$ in a reduced basis is known to be $g_0, ..., g_m, ..., g_t$. We know $g_0 = 1$ always, and a basis element of degree 0 can always be the global 1 function—the function whose value is 1 everywhere. Assume that all the basis elements up to degree $m-1$ have been found. Suppose that $m$ is the next integer such that $g_m \neq 0$. The subspace of $S_m^r(\Delta)$ generated by the module basis elements of degree $m-1$ or less will be $\hat{S} = \{S_{m-1}^r(\Delta) \bigcup_{i=1}^d x_i S_{m-1}^r(\Delta)\}$, where $xM = \{(xm_1, ..., xm_k) \mid (m_1, ..., m_k) \in M\}$. The reduced basis elements of degree $m$ will be the vector space basis elements of the vector space $S_m^r(\Delta)/\hat{S}$.

The space $\hat{S}$ is the null space of the matrix given by the intersection of the row spaces of $d+1$ matrices. Each matrix represents the monomial equations corresponding to the equations $x_k(f_i - f_j - g_{ij} l_{ij}^{r+1})$, where the $f_i$'s are restricted to degree $m-1$ or less. These matrices are each a permutation of the columns of

$$\hat{A} = \begin{pmatrix} A_{m-1} & Z \\ 0 & I \end{pmatrix},$$

where $Z$ is the matrix of all 0's of the same number of rows as $A_{m-1}$ and enough columns to give $\hat{A}$ the same number of columns as $A_m$. Let the intersection of the row spaces of these matrices have basis rows $B = \{b_1, ..., b_k\}$.

**Proposition 2.9.** *The vector space $S_m^r(\Delta)/\hat{S}$ is isomorphic to the null space of $A_m B'$, where $B'$ is the transpose of $B$.*

*Proof.* Suppose that $f \in S_m^r(\Delta)/\hat{S}$; then $f = \sum \lambda_i b_i$. Thus $S_m^r(\Delta)/\hat{S} \cong \{\Delta = (\lambda_1, ..., \lambda_k) \mid A_m B' \Delta' = 0\}$. ∎

Thus $D \subset S^r(\Delta)$ is the set of elements of degree $m$ for some reduced basis if and only if $D$ corresponds to a basis for the null space of $A_m B'$.

**Example.** We continue the previous example with a 2-dimensional complex consisting of two regions that meet along the line $x = 0$. $(f_1, f_2, g)$ is an element in $S^1(\Delta)$ if $f_1 - f_2 + x^2 g = 0$. It is clear by observation that a reduced basis for this module will be $(1, 1, 0)$ and $(0, x^2, 1)$. Consider how the preceeding theory leads us to find the basis element of degree 2.

In the matrix $A_2$ given below, we show the correspondence between the columns of the matrix and the coefficients of the monomials of $f_1$, $f_2$, and $g$. Similarly, we show the correspondence between the rows of the matrix and the restrictions on the coefficients of monomials given by the equation $f_1 - f_2 + x^2 g = 0$:

$$A_2 = \begin{array}{cc} & \begin{array}{ccccccccccccccc} & & f_1 & & & & & & f_2 & & & & & g & \\ 1 & x & y & x^2 & xy & y^2 & 1 & x & y & x^2 & xy & y^2 & 1 & x & y \end{array} \\ \begin{array}{c} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \\ x^3 \\ x^2y \end{array} & \begin{pmatrix} 1 & & & & & & -1 & & & & & & & & \\ & 1 & & & & & & 1 & & & & & & & \\ & & 1 & & & & & & 1 & & & & & & \\ & & & 1 & & & & & & -1 & & & 1 & & \\ & & & & 1 & & & & & & -1 & & & & \\ & & & & & 1 & & & & & & 1 & & & \\ & & & & & & & & & & & & & 1 & \\ & & & & & & & & & & & & & & 1 \end{pmatrix} \end{array}.$$

For convenience, we rewrite $A_2$ as

$$A_2 = \begin{pmatrix} I_6 & -I_6 & & J \\ 0 & 0 & 0 & I_2 \end{pmatrix},$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using the same ordering of the columns, $\hat{A}_1$ is given by

$$\hat{A}_1 = \begin{pmatrix} I_3 & 0 & -I_3 & 0 \\ 0 & I_3 & 0 & 0 \\ 0 & 0 & 0 & I_6 \end{pmatrix}.$$

The intersection of the row spaces of $\hat{A}_1$, $x\hat{A}_1$, and $y\hat{A}_1$ is given by

$$B = \begin{pmatrix} I_6 & -I_6 & 0 \\ 0 & 0 & I_3 \end{pmatrix}.$$

Thus,

$$A_2 B' = \begin{pmatrix} 2I_6 & & J \\ 0 & 0 & I_2 \end{pmatrix}.$$

The basis element for the null space of $A_2 B'$ is $\lambda = (0, 0, 0, 1, 0, 0, -2, 0, 0)$. Thus the new basis element for the null space of $A_2$ is given by

$B^{\prime\prime} = (0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, -2, 0, 0)$, that is, $f_1 = x^2$, $f_2 = x^2$, and $g = -2$. This basis element is a linear combination (over $\mathbb{R}$) of the basis elements observed at the outset.

## 3. Special Case: Graded Modules

In this section we restrict our attention to the case where $S^r(\Delta)$ is a graded module. $R$ is a *graded ring* if there is a family $\{R_n : n > 0\}$ of subgroups of $R$ such that $R = \bigotimes R_n$ and $R_n R_m \subset R_{n+m}$. A *graded R-module M* is an $R$-module such that $M = \bigotimes M_n$, for subgroups $M_n$, with $n \in Z$, and $R_n M_n \subset M_{n+m}$. An element $m \in M_n$ is said to be *homogeneous* of degree $n$. For example if $A = \mathbb{R}[x_1, x_2, ..., x_s]$, then it is in fact a graded ring with $A_n = \langle X^\omega : |\omega| = n \rangle$. In this example the degree of an element in $A_n$ is equal to its degree as a polynomial in the usual sense.

For a graded $k$-algebra $R$, over a field $k$, define the *Krull dimension*, $\dim R$, to be the least number $d$ of homogeneous elements of positive degree, $\theta_1, \theta_2, ..., \theta_d$, such that $R/\langle \theta_1, \theta_2, ..., \theta_d \rangle$ is a finite dimensional $k$-vector space. In this case $\theta_1, \theta_2, ..., \theta_d$ is a *homogeneous system of parameters* for $R$. For a graded $R$-module $M$, define $\dim M$ to be the Krull dimension of $R/(\operatorname{Ann} M)$, where $\operatorname{Ann} M = \{r \in R \mid rM = 0\}$. A sequence $\theta_1, \theta_2, ..., \theta_d$ is a *homogeneous system of parameters* for $M$ if $M/\langle \theta_1, \theta_2, ..., \theta_d \rangle M$ is a finite dimensional vector space over $k$. If $S^r(\Delta)$ is graded, then since $\operatorname{Ann} S^r(\Delta) = \{0\}$ we get that $\dim S^r(\Delta) = \dim A = d$. Also, $(x_1, ..., x_d)$ is a homogeneous system of parameters for $S^r(\Delta)$.

THEOREM 3.1. *Let $M$ be a graded $R$-module such that $M$ is free over $k[\theta_1, \theta_2, ..., \theta_d]$, where $\theta_1, \theta_2, ..., \theta_d$ is a homogeneous system of parameters for $M$ (i.e., $M$ is Cohen–Macaulay). Let $B = \{\zeta_1, \zeta_2, ..., \zeta_s\} \in M$ be a set of homogeneous elements in $M$. Then $B$ is a basis for $M$ over $k[\theta_1, \theta_2, ..., \theta_d]$ if and only if the image of $B$ is a $k$-basis for $M/\langle \theta_1, \theta_2, ..., \theta_d \rangle$.*

*Proof.* See, for example, [14] or [15]. ∎

PROPOSITION 3.2. *If $\Delta$ has only a single interior vertex and all $(d-1)$-faces pass through that vertex, then $S^r(\Delta)$ is a graded $A$-module.*

*Proof.* Without loss of generality (see, for instance, [9 Sect. 1.3]) we may assume that the interior vertex of $\Delta$ is embedded at the origin. Then each hyperplane 1 is homogeneous. Each equation (2.8) is graded with respect to the total degree of monomials. That is, let $f|_m$ be the monomials of $f$ of degree $m$; then $(f_i, f_j, g_{ij})$ satisfies (2.8) implies that

$$f_i|_m - f_j|_m + g_{ij}|_m \, l_{ij}^{r+1} = 0 \ .$$

Thus $(f_1, ..., f_k) \in S^r(\Delta)$ if and only if $(f_1, ..., f_k)|_m \in S^r(\Delta)$, i.e., $S^r(\Delta)$ is graded. ▮

PROPOSITION 3.3. *If $S^r(\Delta)$ is a free graded $A$-module then $S^r(\Delta)$ has a homogeneous module basis.*

*Proof.* This is an application of Theorem 3.1. Consider the homogeneous system of parameters $(x_1, x_2, ..., x_d)$ for $S^r(\Delta)$. $S^r(\Delta)/\langle x_1, x_2, ..., x_d\rangle$, as a graded vector space over $\mathbb{R}$, will have some homogeneous vector space basis. Hence, $S^r(\Delta)$ has a homogeneous module basis. ▮

PROPOSITION 3.4. *A homogeneous free basis $B = \{b_1, b_2, ..., b_s\}$ is always reduced.*

*Proof.* It suffices to consider homogeneous $F \in S^r(\Delta)$. Since $B$ is a basis $F = \sum \lambda_i b_i$, where $\lambda_i \in \mathbb{R}[x_1, x_2, ..., x_d]$. Group the monomials of each $\lambda_i$ such that $\lambda_i = \gamma_i + \delta_i$, where $\deg \gamma_i b_i = \deg F$ or $\gamma_i = 0$, and $\deg \delta_i b_i \neq \deg F$. Now $F = \sum \gamma_i b_i + \sum \delta_i b_i$ implies that $\sum \delta_i b_i = 0$. Since $B$ is a basis, $\delta_i = 0$, $\forall i$. Thus $F = \sum \gamma_i b_i$ and $\deg \gamma_i b_i = \deg F$ or $\gamma_i = 0$. ▮

By Proposition 2.6, if $M$ is a free graded $A$-algebra and if $B$ is a homogeneous basis $B = B_0 \cup B_1 \cup \cdots \cup B_n$, where $B_i = B \cap M_i$, then if $B_i'$ is a basis for $\langle B_i \rangle$, $B_0 \cup B_1 \cup \cdots \cup B_i' \cup \cdots \cup B_n$ is a basis for $M$. In particular, if a graded basis exists, it can be constructed as the union of the vector space bases of the spaces $\hat{M}_i = M_i/\langle M_0 \cup M_1 \cup \cdots \cup M_{i-1}\rangle \cap M_i$ for $i = 0, 1, 2, ....$ For instance, the zeroeth term is simply $M_0$ and the first term is $\hat{M}_1 = M_1/(xM_0 + yM_0)$. If the degrees of the basis elements are known to be $d_1, d_2, ..., d_k$, say, then it is enough to consider the corresponding $\hat{M}_{d_i}$. This simplifies the work involved in trying to construct a reduced module basis, since we need only consider a homogeneous set of equations at each stage.

When $d = 2$ and $\Delta$ has only one interior vertex then $\dim S_m^r(\Delta)$ is given in [12]:

$$\dim S_m^r(\Delta) = \begin{cases} \binom{m+2}{2} & 0 \leqslant m < r \\ \binom{m+1-r}{2} f_1^0 + \binom{r+2}{2} + \sum_{j=1}^{m-r} (r+j+1-je)_+ & m \geqslant r, \end{cases}$$

where $f_1^0 = |\Delta_1^0|$ is the number of interior edges of $\Delta$ and $e$ is the number of different slopes on the interior edges.

In this case the third difference is

$$D^3(m) = \begin{cases} f_1^0 - e & m = r+1 \\ (e-1)\left(1 - \left(\dfrac{1+r}{e-1}\right)_f\right) & m = r+1+\left\lfloor \dfrac{1+r}{e-1} \right\rfloor \\ (e-1)\left(\dfrac{1+r}{e-1}\right)_f & m = r+2+\left\lfloor \dfrac{1+r}{e-1} \right\rfloor \\ 0 & \text{otherwise,} \end{cases} \tag{3.5}$$

where $\lfloor p \rfloor$ is the integer part of $p$ and $(p)_f = p - \lfloor p \rfloor$. From (2.3), $g_i = D^3(i)$ in this case. For example, for $\Delta$ with one interior vertex and four interior edges each with a different slope, the degrees of the basis elements have been given in Table I. In Table I $i$ is the degree of a basis element and $r$ is the degree of smoothness.

The calculation of (3.5) shows that for $\Delta$ with only one interior vertex there will be only two values of $i > r+1$ such that $g_i$ is not 0. Each basis element of degree $r+1$ corresponds to a pair of $(d-1)$-faces on the same hyperplane. We can let the element be the function which is 0 on one side of the hyperplane and $l^{r+1}$ on the other. Thus to find a basis using vector space techniques we need only consider the two systems of equations for degree $> r+1$.

## 4. Cross-Cut Grids

Much work has been done towards the computation of the dimension and the construction of bases for the class of multivariate splines on cross-

TABLE I

$g_i$ for $\Delta$ with One Interior Vertex and Four Interior Edges

| $r \backslash i$ | 0 | 1 | 2 | 3 | 4 | 5 | . . . . . | $r+1+\left\lfloor \frac{1+r}{3} \right\rfloor$ | $r+2+\left\lfloor \frac{1+r}{3} \right\rfloor$ |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 1 | | | | | | |
| 1 | 1 | | 1 | 2 | | | | | |
| 2 | 1 | | | 3 | 0 | | | | |
| 3 | 1 | | | | 2 | 1 | | | |
| 4 | 1 | | | | | 1 | 2 | | |
| . | . | | | | | | . | | |
| . | . | | | | | | . | | |
| . | . | | | | | | . | | |
| $r$ | 1 | | | | | | | $3\left(1\ \left(\frac{1+r}{3}\right)_f\right)$ | $3\left(\frac{1+r}{3}\right)_f$ |

cut grid partitions. In [7, 8] vector space bases for some classes of splines on certain cross-cut grid partitions are discussed. This section extends that work to module bases for $S^r(\Delta)$, when $\Delta$ is a cross-cut grid partition of a disk. The main part of this section shows how finding bases for disks with one interior vertex (as done in the previous section) comprises all the work for finding a basis for $S^r(\Delta)$ when $\Delta$ is a cross-cut grid in $\mathbb{R}^2$.

We develop a method for finding a basis for $S^r(\Delta)$ when $\Delta$ is a cross-cut grid on a 2-dimensional disk. The process assumes that we can find bases for certain local subcomplexes. Given a domain $D \subset \mathbb{R}^d$ and two complexes $\Sigma$ and $\Pi$ on $D$, $\Sigma$ is a *coarser* subdivision than $\Pi$ and $\Pi$ is a *finer* subdivision than $\Sigma$, if (considered as subsets of $D$) whenever $\sigma$ is a $d$-face of $\Pi$ then there is a $d$-face $\tau$ of $\sum$ such that $\tau \supset \sigma$.

PROPOSITION 4.1. *Given a domain $D \subset \mathbb{R}^d$ and two complexes $\Sigma$ and $\Pi$ on $D$ such that $\Sigma$ is a coarser subdivision than $\Pi$, if $f: D \to \mathbb{R}$ is such that $f \in S^r(\Sigma)$, then $f \in S^r(\Pi)$.*

*Proof.* Any $d$-face $\sigma$ of $\Pi$ is contained in a $d$-face $\tau$ of $\Sigma$, so that $f|_\sigma = f|_\tau \in \mathbb{R}[x_1, ..., x_d]$. Further, $f \in S^r(\Sigma)$ means that $f \in S^r(D)$, i.e., $f$ is $C^r$, so that $f \in S^r(\Pi)$. ∎

Call a complex $\Sigma$ a *star complex* if either $\Sigma$ has only one interior vertex $v$ and all interior edges of $\Sigma$ are incident to $v$ or $\Sigma$ has no interior vertices. Given a cross-cut grid $\Delta$ on a domain $D$ and any vertex $v \in \Delta$, we consider a certain star complex $\Sigma_v$ on $D$ which is a coarser subdivision on $D$ then $\Delta$. By Proposition 4.1, if $f \in S^r(\Sigma_v)$ then $f \in S^r(\Delta)$. Given a spline $f$ on $\Delta$, a 2-face $\sigma \in \Delta$ is in the *support* of $f$ if $f|_\sigma \neq 0$. The vertices of $\Delta$ can be ordered in such a way that elements from each star basis can be used to reduce the number of support faces.

Inductively define a total order on $\mathbb{R}^d$ to be *admissible* as follows. For every point $p$ in $\mathbb{R}^d$ there exists a hyperplane such that every point on one half space given by the hyperplane is greater than $p$ and every point on the other side of the hyperplane is less than $p$. Further, the induced ordering on the hyperplane is admissible in $\mathbb{R}^{d-1}$. The unique ordering on $\mathbb{R}^0$ is admissible. The usual orderings, lexicographic and graded lexicographic, are both admissible orders.

Given a cross-cut grid $\Delta$ on a disk $D$, order the vertices of $\Delta$ using any admissible ordering on $\mathbb{R}^2$. Say the ordered vertices are $v_1, v_2, ..., v_n$. For each $v_j$, associate a star complex $\Sigma_j$ of $v_j$ defined as follows: $v_i \in \Sigma_j$ if and only if $i = j$ or $v_i$ is on the boundary of $\Delta$. For $i \neq j \neq k$, $(v_i, v_k) \in \Sigma_j$ if $(v_i, v_k)$ is on the boundary of $\Delta$. $(v_i, v_j) \in \Sigma_j$ if $i \geqslant j$ and there exists a cross-cut hyperplane $l$ of $\Delta$ such that both $v_i$ and $v_j$ lie on $l$. See Fig. 2 for examples. In Fig. 2 lexicographic order has been used with $x > y$.

Define the *least vertex* of a 2-face $\sigma$ to be the least vertex $v$ with respect

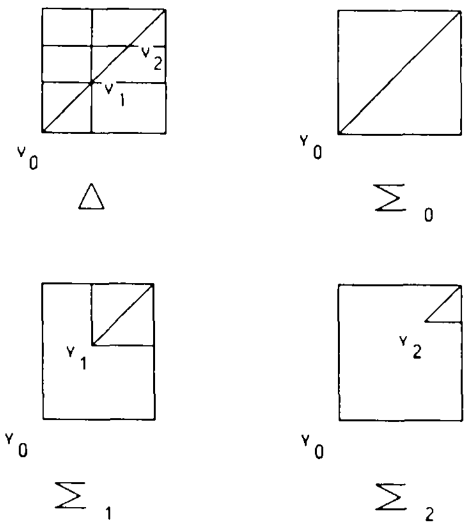


FIG. 2. $\Delta$ and some of its associated complexes $\Sigma_i$.

to the ordering $\geqslant$ such that $v$ is a vertex of $\sigma$. Since $\geqslant$ is a total ordering such a vertex is always uniquely determined.

In each $\Sigma_j$ defined above, all but perhaps one 2-face of $\Sigma_j$ will have $v_j$ as least vertex. Let $\sigma_j$ be the 2-face with least vertex not equal to $v_j$ (if no such 2-face exists let $\sigma_j$ be the empty set). The face $\sigma_j$ may be all of $D$. See Fig. 2 for examples. For each star complex $\Sigma_j$ such that $\sigma_j \neq D$ construct a basis $B_j$ for the complex with the following properties. The global constant function $\hat{1}$ is an element of the basis, where $\hat{1}$ is defined by $\hat{1}|_\sigma = 1$ for all $\sigma \in \Sigma_j$. If $b \in B_j$ and $b \neq \hat{1}$, then the support of $b$ does not contain $\sigma_j$; i.e., $b|_{\sigma_j} = 0$. By Proposition 4.1 the basis elements of $B_j$ can be considered as functions in $S^r(\Delta)$.

THEOREM 4.2. *$\mathscr{B} = [\bigcup(B_j - \{\hat{1}\}) \cup \{\hat{1}\}]$ is a basis for $S^r(\Delta)$, where the union is taken over $J = \{j : v_j$ is a vertex of $\Delta$ and $\sigma_j \neq D\}$.*

*Proof.* For each vertex $v_j$ we get that $|B_j - \{\hat{1}\}|$ is equal to the number of 2-faces for which $v_j$ is the least vertex. The number of 2-faces in $\Sigma_j$ for which $v_j$ is the least vertex is equal to the number of 2-faces in $\Delta$ for which it is the least vertex. Hence, $|\mathscr{B}| = f_d$. Since it is known that rank of $S^r(\Delta)$ is $f_d$ [11] if $\mathscr{B}$ spans then it is a module basis. To see that $\mathscr{B}$ spans consider any $f \in S^r(\Delta)$. The proof is by induction on the support of an element $f$. Let $v_{j^*}$ be the lowest vertex (with respect to $\leqslant$) such that $v_{j^*}$ is the least vertex of some 2-face in the support of $f$. Assume first that $j^*$ is the largest element in $J$. In this case $f$ is in the span of the set $B_{j^*} - \{\hat{1}\}$.

In general let $f'$ be the restriction of $f$ to $\text{star}_\Delta(v_{j^*})$. By the choice of $v_{j^*}$, it must be that $f'|_{\sigma_{j^*}}=0$. Therefore we can consider $f'$ as a function on $\Sigma_{j^*}$. Further, $f'$ is in the span of $B_{j^*}-\{\hat{1}\}$. So it is sufficient to show $f-f'$ is spanned by $\mathscr{B}$. By construction, the least vertex of any 2-face in the support of $f-f'$ will be greater than $v_{j^*}$. Hence, by induction, $f-f'$ is in the span of $\mathscr{B}$ and thus $f$ is as well. ∎

The construction of a basis of $S^r(\Delta)$ from bases for certain $S^r(\Sigma)$ provides a means to calculate the dimension of $S^r(\Delta)$ from the dimensions of $S^r(\Sigma)$. We have

$$\dim S^r_m(\Delta)=\sum_{v\in\Delta_0}\left(\dim S^r_m(\Sigma_v)-\binom{m+2}{2}\right)+\binom{m+2}{2},$$

where $\binom{m+2}{2}$ is the dimension of the global space. Equation (3.5) gives the dimension of $S^r_m(\Sigma_v)$, for $v$ an interior vertex. When $v$ is a boundary vertex and there are $e_v$ interior edges then $\Sigma_j$ is stacked. Thus in this case the dimension of $S^r_m(\Sigma_j)$ is given by (2.5). Since every interior edge of $\Delta$ is an interior edge in exactly one star complex $\Sigma$, we have the following result.

THEOREM 4.3 [8, Theorem 3.1]. *For a cross-cut grid $\Delta$ on domain $D\subset\mathbb{R}^2$ that is a disk*, $\dim S^r_m(\Delta)=\binom{m+2}{2}$ *for $m\leqslant r$ and for $m\geqslant r+1$*,

$$\dim S^r_m(\Delta)=\binom{m+1-r}{2}f^o_1+\left(\binom{r+2}{2}-\binom{m+2}{2}\right)f^o_0$$
$$+\sum_{v\in\Delta^o_0}\sum_{j=1}^{m+r}(r+j+1-je_v)_+.$$

It was shown by Shumaker in [12] that this number is a lower bound for $\dim(S^r_m(\Delta))$ for all $\Delta\subset\mathbb{R}^2$. He also showed this to be the dimension in certain cross-cut cases. Chui and Wang [7] showed this to be the dimension of cross-cut grid spaces via vector space analysis.

The number of basis elements of each degree $(g_i(\Delta))$ for a reduced module basis of $\Delta$ can be computed either by taking differences of the above formula for $\dim S^r_m(\Delta)$ or by summing the $g_i(\Sigma_v)$ over the vertices of $\Delta$ which were used in the consltruction of a basis for $\Delta$. Note that while the ordering of the vertices of $\Delta$ affects the basis constructed, since $\dim S^r_m(\Delta)$ is invariant, the $g_i(\Delta)$ are invariant as well. The polynomial $p(t)=\Sigma g_i t^i$ is just the sum of the polynomials corresponding to the star complexes.

A cross-cut grid partition $\Delta$ in $d$-space is *simple* if every vertex is the intersection of exactly $d$ hyperplanes (or equivalently if every $(d-k)$-face is the intesection of exactly $k$ hyperplanes). All 1-dimensional connected complexes are simple cross-cuts. In 2-dimensional complexes a simple cross-cut has exactly two hyperplanes meeting at any interior vertex. In [7] Chui and Wang give a vector space basis for $S_m^r(\Delta)$ for simple cross-cut when $d=2$ and comment that it is easy to find such a basis for general $d$. A module basis for this case is presented below.

Let $\Delta$ be a simple cross-cut grid on a domain $D$ in $\mathbb{R}^d$, with cross-cut hyperplanes $l_1, l_2, ..., l_n$. For each cross-cut $l_i$, pick a "positive" side. Define the piecewise polynomial $B_i$ on $\Delta$ by

$$B_i(x_1, ..., x_d) = \begin{cases} l_i^{r+1} & \text{if } (x_1, ..., x_d) \text{ is on the positive side of } l_i \\ 0 & \text{otherwise.} \end{cases}$$

For each set $t=\{i_1, i_2, ..., i_k\}$ such that $\tau_t = l_{i_1} \cap l_{i_2} \cap \cdots \cap l_{i_k} \cap D \neq \varnothing$, define the piecewise polynomial $B_t$ on $\Delta$ by

$$B_t = \prod_{i \in t} B_i.$$

The $B_t$ are basis elements for the star complexes of $\Delta$. If $|t|=d$ then $B_t$ is a basis element for the star complex of the interior vertex at $\tau_t$. If $|t|<d$ then $B_t$ is a basis element for a boundary vertex on $\tau_t$.

THEOREM 4.4. *The set of $B_t$ and the global function $\hat{1}$ on $\Delta$ together are a basis for $S_m^r(\Delta)$.*

*Proof.* This proof is analogous to the proof of 4.2. ∎

The number of basis elements of degree $k(r+1)$, denoted $g_{(kr+k)}$, is given by the number of sets $t$ such that $|t|=d-k$ and $\tau_t \neq \varnothing$. For instance, $g_{r+1}=n$, the number of hyperplanes that cross $\Delta$. Define $L_k$ to be the number of sets $t$ such that $|t|=d-k$ and $\tau_t \neq \varnothing$. By Eq. (2.1),

$$\dim S_m^r(\Delta) = \sum_{k(r+1) \leqslant m} L_k \binom{m-kr-k+d}{d}. \tag{4.5}$$

For example, we consider the special case given by Chui and Wang in [7]. In this case let $D$ be a domain bounded by hyperplanes normal to the axes and let each of the cross-cut hyperplanes also be normal to some axis, i.e., each hyperplane is given by an equation of the form $x_i=\lambda$ for some $i=1, ..., d$ and some $\lambda \in \mathbb{R}$. For each $i$, let $c_i$ be the number of hyperplanes that cross the interior which are parallel to the hyperplane $x_i=0$. Then $L_1 = c_1 + c_2 + \cdots + c_d$, $L_d = c_1 c_2 \cdots c_d$, and in general

$$L_k = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant d} c_{i_1} c_{i_2} \cdots c_{i_k}.$$

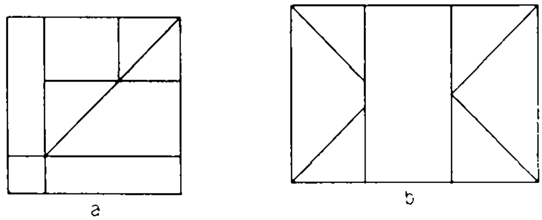


FIG. 3. Quasi-cross-cut partitions.

A complex $\Delta$, on a domain $D$ in 2-dimensions, is a *quasi-cross-cut* complex if for any line $l$ which is the support of an edge $\sigma$ of $\Delta$, there is some point $(x_0, y_0) \in l \cap D$ with the following property. Say $D_+ = \{(x, y) \mid (x, y) > (x_0, y_0)\}$ and $D_- = \{(x, y) \mid (x, y) < (x_0, y_0)\}$ such that either $l \cap D_+ \subset \Delta_1^0$ or $l \cap D_- \subset \Delta_1^0$, or both. That is, each edge lies on either a cross-cut or a half line that begins in $D$; see Fig. 3 for examples. In [8] it is stated that the dimension given for cross-cut partitions in two dimensions, (4.3), also holds for quasi-cross-cuts.

In the following situation there is a basis for $S^r(\Delta)$. Suppose that there is a total ordering on the vertices of $\Delta$ such that for each vertex $v$, any edge $e$ incident to $v$ lies on a quasi-cross-cut $l$ such that the associated point $(x_0, y_0)$ is less than $v$. That is, every edge extends in the positive direction from each vertex on which it is incident. In Fig. 3a lexicographic order with $x < y$ is such an order. In Fig. 3b there is no such order. If there is such an ordering then a basis for $S^r(\Delta)$ can be construced as in the cross-cut case. The proof of this is exactly the same as the proof of the cross-cut case.

## 5. Reduced Generating Sets and Grobner Bases

Many of the ideas of this paper can be used even if there is not a reduced free basis for the module $S^r(\Delta)$. A generating set $\{b_i\}$, for a module $M$, over the polynomial ring, is *reduced* if for any $m \in M$ there exists a representation $m = \Sigma \lambda_i b_i$, such that $\deg \lambda_i b_i \leqslant \deg m$, for all $i$ such that $\lambda_i \neq 0$. Given a generating set, there are relations between the generators, i.e., polynomials $p_1, ..., p_k$, such that $\Sigma p_i b_i = 0$. We define the degree of this relation to be the degree of the highest monomial of $\Sigma p_i b_i$. There exists a reduced generating set for this module of relations. Further, there may be relations between the relations, etc. Given such a reduced generating set define $g_{0i}$ to be the number of generating elements of degree $i$, $g_{1i}$ to be the number of relations between the generators of degree $i$, $g_{2i}$ to be the number of relations between relations of degree $i$, etc. The results of

Section 2 extend to the case when $S^r(\Delta)$ has a reduced generating set. In this case,

$$\dim S^r_m(\Delta) = \sum_{j=0}^{m} \binom{m-j+d}{d} \sum_{i=0}^{m} (-1)^i g_{ij}$$

and so

$$D^{d+1}(m) = \sum_{i=0}^{m} (-1)^i g_{im}.$$

An alternative approach to finding reduced bases for any module is via the technique of Grobner Bases. This is an algorithm which can find a reduced generating set and the resolution of the module (the generating set and the relations between the generators and the relations between the relations etc.). This algorithm can always be used to find a reduced generating set for $S^r(\Delta)$; however, it may not lead to a reduced basis even when one exists. For a detailed discussion on the use of the Grobner Basis Theory and its applications to the module $S^r(\Delta)$, see [9, 11].

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